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# Fermi Coordinates of an Observer Moving in a Circle in Minkowski Space: Apparent Behavior of Clocks

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Coordinate transformations are derived from global Minkowski coordinates to the Fermi coordinates of an observer moving in a circle in Minkowski space-time. The metric for the Fermi coordinates is calculated directly from the tensor transformation rule. The behavior of ideal clocks is examined from the observer's reference frame using the Fermi coordinates. A complicated relation exists between Fermi coordinate time and proper time on stationary clocks (in the Fermi frame) and between proper time on satellite clocks that orbit the observer. An orbital Sagnac-like effect exists for portable clocks that orbit the Fermi coordinate origin. The coordinate speed of light is isotropic but varies with Fermi coordinate position and time. The magnitudes of these kinematic effects are computed for parameters that are relevant to the Global Positioning System (GPS) and are found to be small; however, for future high-accuracy time transfer systems, these effects may be of significant magnitude.

## I. INTRODUCTION

During the past several decades there have been significant advances in the accuracy of satellite-based navigation and time transfer systems [1]. The most recent example of such a system is the Global Positioning System (GPS), which is run by the U. S. Department of Defense [2-4]. The GPS is based on a network of 24 atomic clocks in space, from which a user can receive signals and compute his position and time. The order of magnitude of the accuracy of the GPS currently is 10 m and 20 ns [5]. Future navigation and time transfer systems are being contemplated that will have several orders of magnitude better position accuracy and subnanosecond time accuracy [6-8].

Over satellite to ground distances, precise measurements must be interpreted within the framework of a relativistic theory [9]. The design of high-accuracy navigation and time transfer systems depends on detailed modeling of the measurement process. The comparison of experiment with theory in general relativity is more subtle than in special relativity or other areas of physics for two reasons. First, due to the possibility of making coordinate gauge transformations, it is often not clear what is a measurable quantity and what is an artifact of the coordinate system [10,11]. Second, all measurements must in principle be interpreted in the comoving frame of the experimental apparatus [12]. In the case of the GPS, even theoretical relativistic treatments do not interpret the measured observable in the comoving frame of the apparatus, (see, for example, Schwarze et al. [13]).

The principal reason for investigating relativistic effects in detail is to improve the current accuracy of GPS and to create future time transfer and navigation systems that have several orders of magnitude better accuracy. At the present time, it is wellknown that small anomalies exist in position and time computed from GPS data. The origin of these anomalies is not understood. In particular, GPS time transfer data from the U. S. Naval Observatory indicates that GPS time is periodic with respect to the Master Clock, which is the most accurate source of official time for the U. S. Department of Defense [14-16]. Furthermore, other anomalies have been found in Air Force monitor station data that are not understood at present [17]. The work below is a first step in revisiting from a first principles approach some of the physics that has gone into the design of the GPS.

A theoretical framework known as the tetrad formalism has been developed for relating measurements to relativistic theory [12,18-20]. However, due to a historical lack of accurate measurements over satellite to ground distances, in most cases it has not been necessary to apply the tetrad formalism in the area of satellite-based systems. This situation is rapidly changing due to improved measurement technology [9,23-25].

In the tetrad formalism, measurements are interpreted as projections of tensors on the tetrad basis vectors. Consequently, these projections are invariant quantities (scalars) under transformation of the space-time coordinates. However, these projections do depend on the world line of the observer and the choice of local Cartesian axes used by the observer [20,12]. The need for the tetrad formalism to relate experiment to theory, as well as the problem of measurable quantities in general relativity, is well discussed by Synge [12], Soffel [21], and Brumberg [22].

The tetrad formalism was initially investigated for the case of inertial observers, which move on geodesics [18–20,26–28]. However, many users of geodetic satellite systems are terrestrially based observers, or are based in non-inertial platforms, such as missiles. The theory for the case of non-inertial observers has been investigated by Synge [12], who considered the case of non-rotating observers moving along a time-like world line, and by others [29–33], who considered accelerated, rotating observers. For arbitrary observer motions, the effects are indeed very complicated, and the general theory gives limited physical insight. For this reason, in this work, I work out in detail a simple model problem. I treat the particular case of an observer moving in a circle in Minkowski space. I use the tetrad formalism [12] to obtain the transformation from an inertial frame in Minkowski space to the accelerated frame of reference of an observer moving in a circle. The observer Fermi-Walker transports his tetrad basis vectors. The case presented here corresponds to an observer kept in a circular trajectory by rockets: i.e., he is not on a geodesic. The explicit results presented below form a particular case of the general results obtained by Synge [12], specialized to zero gravitational field and circular motion. The situation treated here also corresponds to a reference frame that has a constant orientation with respect to the distant stars [34], such as an Earth-centered inertial reference frame used in GPS satellite computations. The results are given here as expansions in small velocity of the observer, compared with speed of light. However, the coordinate transformation equations are not expanded in a power series in time, and hence they are valid for arbitrarily long times.

This work explores the apparent behavior of ideal clocks as observed from a Fermi coordinate frame of reference of an observer moving in a circle in flat space-time. The Fermi coordinate frame is the closest type of non-rotating frame to that used in experiments and in GPS. In particular, a Fermi coordinate frame is a non-rotating frame of reference, where Coriolis forces are absent, and light travels essentially along a straight line [12]. The observer's Fermi frame in this work is analogous to the Earth-centered inertial (ECI) frame used in GPS. The ECI frame moves with the Earth but maintains its orientation with respect to the distant stars.

This work neglects all gravitational field effects that are significant for the actual behavior of GPS clocks. As such, this work treats only the kinematic effects associated with the circular motion of the Fermi coordinate frame about a central point (analogous to the sun's position). Since gravitational fields are neglected in this work, the observer's tetrad is Fermi-Walker transported along a time-like world line (a circle in 3-d space). In the case of GPS, the origin of the ECI frame follows a geodesic path (the Earth's orbit). Therefore, the observer's Fermi frame in this work is an accelerated, non-rotating frame, while the ECI frame is in free fall. As remarked above, the effects on clocks are in general very complicated, even without the gravitational field. For this reason, I work out in detail the specific case of an observer in circular motion in Minkowski space-time, and I apply the results to parameters that are relevant to the GPS.

## II. FERMI-WALKER TRANSPORT DIFFERENTIAL EQUATIONS

Consider a time-like world line  $C$  of an observer moving in a circle in 3-dimensional (3-d) space given by

$$x^0 = x_0^0 + u \quad (1)$$

$$x^1 = x_0^1 + a \cos(\omega u/c) \quad (2)$$

$$x^2 = x_0^2 + a \sin(\omega u/c) \quad (3)$$

$$x^3 = x_0^3 \quad (4)$$

where  $x_0^0$ ,  $x_0^1$ ,  $x_0^2$ , and  $x_0^3$  are constants,  $u$  is a parameter along the world line, and  $c$  is the speed of light in an inertial frame in vacuum. The observer travels in a circle in the  $x$ - $y$  plane (see Figure 1). Along his world line, the observer travels with a four velocity

$$u^i = \frac{dx^i}{ds} \quad (5)$$

and has an acceleration

$$w^i = \frac{\delta u^i}{\delta s} = \frac{du^i}{ds} + \Gamma_{jk}^i u^j u^k \quad (6)$$

where  $\Gamma_{jk}^i$  is the affine connection. From the normalization of the 4-velocity,  $u^i u_i = -1$ , I obtain the relation between the arc length,  $s = c\tau$ , where  $\tau$  is the proper time, and the parameter  $u$ ,

$$u = \gamma s \quad (7)$$

where  $\gamma = (1 - \nu^2)^{-1/2}$ . The parameter  $\nu$  is the dimensionless velocity, given by

$$\nu = \frac{a\omega}{c} \quad (8)$$

I use the convention that Roman indices take the values  $i = 0, 1, 2, 3$  and Greek indices take values  $\alpha = 1, 2, 3$  (see the Appendix for the conventions used in this work).

In Minkowski space, the affine connection components all vanish:  $\Gamma_{jk}^i = 0$ . Hence, the explicit expressions for the 4-velocity and acceleration are

$$u^i(s) = \gamma \left( 1, -\frac{a\omega}{c} \sin\left(\frac{\gamma\omega s}{c}\right), \frac{a\omega}{c} \cos\left(\frac{\gamma\omega s}{c}\right), 0 \right) \quad (9)$$

$$w^i = -a\gamma^2 \left( \frac{\omega}{c} \right)^2 \left( 0, \cos\left(\frac{\gamma\omega}{c}s\right), \sin\left(\frac{\gamma\omega}{c}s\right), 0 \right) \quad (10)$$

As the observer moves on the time-like world line  $C$ , he carries with him an ideal clock and three gyroscopes. I use the notation  $x^i$  for Minkowski space coordinates. At some initial coordinate time  $x^0$ , the observer is at point  $P_0$  at proper time  $\tau = 0$ , corresponding to  $s = 0$ . On his world line, the observer carries with him three orthonormal basis vectors,  $\lambda_{(\alpha)}^{(i)}$ , where  $\alpha = 1, 2, 3$ , labels the vectors and  $i = 0, 1, 2, 3$  labels the components of these vectors in Minkowski coordinates. These vectors form the basis for his measurements [20] (see Figure 2). The orientation of each basis vector is held fixed with respect to each of the gyroscopes' axis of rotation [35]. The fourth basis vector is taken to be the observer 4-velocity,  $\lambda_{(0)}^{(i)} = v^i$ . The four unit vectors  $\lambda_{(a)}^{(i)}$  form our tetrad [12], an orthonormal set at  $P_0$ , which satisfies

$$g_{ij} \lambda_{(a)}^i \lambda_{(b)}^j = \delta_{ab} \quad (11)$$

where the  $\delta_{ij} = 1$  if  $i = j$ , and zero otherwise.

At a later time  $s = c\tau$ , the observer is at a point  $P$ . The observer's orthonormal set of basis vectors are related to his tetrad basis at  $P_0$  by Fermi-Walker transport. Fermi-Walker transport preserves the lengths and relative angles of the transported vectors. Furthermore, Fermi-Walker transport of basis vectors along the observer's world line corresponds to keeping the vectors as parallel as possible by keeping their orientation constant relative to the spin axes of the gyroscopes [35]. For a given vector, with contravariant components  $f^i$  in Minkowski space, its components at  $P$  are related to its components at  $P_0$  by the Fermi-Walker transport differential equations [12]

$$\frac{\delta f^i}{\delta s} = W^{ij} f_j \quad (12)$$

where

$$W^{ij} = u^i w^j - w^i u^j \quad (13)$$

When we use Eq. (13) to transport a vector  $f^i$  that is orthogonal to the 4-velocity,  $u^i f_i = 0$ , the second term in Eq. (13) does not contribute. We refer to transport of such space-like basis vectors as Fermi transport, and  $W^{ij} \rightarrow \tilde{W}^{ij} = u^i w^j$ . For an arbitrary vector  $f^i$  perpendicular to the 4-velocity, the explicit form of the Fermi transport differential equations for the world line in Eqs. (1)–(4) is

$$\frac{df^0}{d\xi} = -\gamma^3 \nu \cos(\gamma\xi) f^1 - \gamma^3 \nu \sin(\gamma\xi) f^2 \quad (14)$$

$$\frac{df^1}{d\xi} = \gamma^3 \nu^2 \cos(\gamma\xi) \sin(\gamma\xi) f^1 + \gamma^3 \nu^2 \sin^2(\gamma\xi) f^2 \quad (15)$$

$$\frac{df^2}{d\xi} = -\gamma^3 \nu^2 \cos^2(\gamma\xi) f^1 - \gamma^3 \nu^2 \cos(\gamma\xi) \sin(\gamma\xi) f^2 \quad (16)$$

$$\frac{df^3}{d\xi} = 0 \quad (17)$$

where the components  $f^i$  are functions of  $\xi$  and the dimensionless proper time is given by

$$\xi = \frac{\omega}{c}s \quad (18)$$

The differential equations in Eqs. (14)-(17) are nothing more than the equations that describe the Thomas precession of an electron's spin vector as it moves in a circular orbit around the nucleus [36]. The solution of Eq. (14)-(17) is given by [35]

$$f^0 = -\gamma\nu A \cos(\gamma^2\xi + \alpha) + \beta \quad (19)$$

$$f^1 = A \cos(\gamma\xi) \cos(\gamma^2\xi + \alpha) + A\gamma \sin(\gamma\xi) \sin(\gamma^2\xi + \alpha) \quad (20)$$

$$f^2 = A \sin(\gamma\xi) \cos(\gamma^2\xi + \alpha) - A\gamma \cos(\gamma\xi) \sin(\gamma^2\xi + \alpha) \quad (21)$$

$$f^3 = \delta \quad (22)$$

where  $A$ ,  $\alpha$ ,  $\delta$  and  $\beta$  are integration constants.

### III. DETERMINATION OF THE TETRAD

At the point  $P_0$ , the tetrad basis vectors satisfy Eq. (11). These equations constitute 12 relations (since Eq. (11) is symmetric in  $\alpha$  and  $\beta$ ) for 16 components  $\lambda_{(a)}^i$ . However, the zeroth member of the tetrad is the 4-velocity, which is assumed known:

$$\lambda_{(0)}^i = u^i = \gamma(1, v^1, v^2, v^3) \quad (23)$$

and is given by Eq. (9). Equation (11) can be rewritten as two equations:

$$g_{ij}\lambda_{(a)}^i\lambda_{(0)}^j = 0, \quad a = 1, 2, 3 \quad (24)$$

$$g_{ij}\lambda_{(\alpha)}^i\lambda_{(\beta)}^j = \delta_{\alpha\beta}, \quad \alpha = 1, 2, 3 \quad (25)$$

Substituting the explicit form of the Minkowski metric (see the Appendix) into Eq. (24), gives a condition on the zeroth components of a tetrad vector if the spatial components are known (Greek indices take values  $\alpha, \kappa = 1, 2, 3$ ):

$$\lambda_{(\alpha)}^0 = \delta_{\beta\kappa} v^\beta \lambda_{(\alpha)}^\kappa \quad (26)$$

Then use of Eq. (26) to eliminate the time components  $\lambda_{(\alpha)}^0$  from Eq. (25) leads to an orthogonality relation for only the spatial components of the tetrad vectors

$$\delta_{\mu\nu}\lambda_{(\alpha)}^\mu\lambda_{(\beta)}^\nu - \delta_{\mu\nu}\delta_{\kappa\gamma}v^\mu v^\kappa\lambda_{(\alpha)}^\nu\lambda_{(\beta)}^\gamma = \delta_{\alpha\beta} \quad (27)$$

The general idea is to solve Eq. (27) for the spatial components of the tetrad at  $s = 0$ , and then to substitute the spatial components into Eq. (26) to obtain the time components of each tetrad vector. Thus, I obtain the tetrad vectors at the initial time corresponding to  $s = 0$ . For each tetrad vector, these components are then used as initial conditions in Eq. (22) to determine the tetrad components at point  $P$  at finite  $s$ .

The exact solution of Eq. (27) can be obtained by noting that for  $v^\alpha = 0$ , the spatial components of the tetrad vectors are orthonormal, and the solution is given by

$$\lambda_{(\alpha)}^\mu = \delta_\alpha^\mu \quad (28)$$

I assume that the tetrad unit vectors are oriented approximately parallel to the observer's local Cartesian  $x$ ,  $y$  and  $z$  axes. For small  $v^\alpha \ll 1$ , the solution of Eq. (27) can be obtained as a triple power series in  $v^\alpha$ , by iteration, starting with Eq. (28) as the first approximation. Having solved for the tetrad components as a power series, it is easy to guess the exact solution to be

$$\lambda_{(\alpha)}^0 = \gamma\delta_{\alpha\beta}v^\beta \quad (29)$$

$$\lambda_{(\alpha)}^\mu = \delta_\alpha^\mu + \frac{\gamma-1}{\nu^2}\delta_{\kappa\alpha}v^\kappa v^\mu \quad (30)$$

$$\lambda_{(0)}^i = u^i = \gamma(1, v^1, v^2, v^3) \quad (31)$$

where the components of dimensionless velocity  $v^\alpha$  are given in Eq. (9).

Equations (29)–(31) give the components of the observer's tetrad basis vectors at point  $P_0$ . At point  $P$ , the arc length is  $s > 0$ , and Eqs. (29)–(31) give the initial components for each tetrad vector. These initial components are used in the general solution of a Fermi-Walker transported vector, given in Eq. (22). The tetrad components at point  $P$  at finite  $s$  are given by

$$\lambda_{(0)}^i = \{\gamma, -\gamma\nu \sin(\gamma\xi), \gamma\nu \cos(\gamma\xi), 0\} \quad (32)$$

$$\lambda_{(1)}^i = \{-\gamma\nu \sin(\gamma^2\xi), \cos(\gamma\xi) \cos(\gamma^2\xi) + \gamma \sin(\gamma\xi) \sin(\gamma^2\xi), \cos(\gamma^2\xi) \sin(\gamma\xi) - \gamma \cos(\gamma\xi) \sin(\gamma^2\xi), 0\} \quad (33)$$

$$\lambda_{(2)}^i = \{\gamma\nu \cos(\gamma^2\xi), -\gamma \cos(\gamma^2\xi) \sin(\gamma\xi) + \cos(\gamma\xi) \sin(\gamma^2\xi), \gamma \cos(\gamma\xi) \cos(\gamma^2\xi) + \sin(\gamma\xi) \sin(\gamma^2\xi), 0\} \quad (34)$$

$$\lambda_{(3)}^i = \{0, 0, 0, 1\} \quad (35)$$

Equations (32)–(35) are valid for all times  $s$  because they are exact solutions of Eqs. (14)–(17) and satisfy the orthogonality relations in Eq. (11).

#### IV. FERMI COORDINATES

Fermi coordinates are defined by the geometric construction shown in Figure 2. Every event  $P'$  in space-time has coordinates  $x^{i'}$  in the global inertial coordinate system. According to the observer moving in circular motion along a time-like world line, the same event has the Fermi coordinates  $X^{(a)}$ ,  $a = 0, 1, 2, 3$ . The first Fermi coordinate,  $X^{(0)} = s$ , is just the proper time (in units of length) associated with the event  $P'$ . The proper time for  $P'$  is defined as the value of arc length  $s$  in space-time at which a space-like geodesic from point  $P$  (which defines  $s$ ) passes through event  $P'$ . The definition of  $s$  requires the tangent vector of this space-like geodesic,  $\mu^i$ , be orthogonal to the observer 4-velocity at  $P$ ,

$$\mu^i v_i|_P = 0 \quad (36)$$

For the simple case of Minkowski space, the geodesic is simply a straight line connecting the points  $P$  and  $P'$

$$\mu^i = N(x^{i'} - x^i(\gamma s)) \quad (37)$$

where  $x^i(u)$  and  $x^{i'}$  are the coordinates of  $P$  and  $P'$ , respectively, in the global inertial frame, and  $N$  is a normalization constant that makes  $\mu^i$  a unit vector. The orthogonality condition in Eq. (36) is

$$g_{ij} \mu^i(s) \lambda_{(0)}^j(s) = 0 \quad (38)$$

and gives an implicit equation for  $s$  for a given  $P'$ . This orthogonality condition gives the first Fermi coordinate of the point  $P'$

$$X^{(0)} = s \quad (39)$$

and Eq. (38) gives  $s$  as an implicit equation:

$$\gamma s = x^{0'} - x_0^0 + \nu \left[ (x^{1'} - x_0^1) \sin\left(\frac{\gamma\omega s}{c}\right) - (x^{2'} - x_0^2) \cos\left(\frac{\gamma\omega s}{c}\right) \right] \quad (40)$$

In the limit of small speeds of the observer,  $\nu \rightarrow 0$ , Eq. (40) gives  $s = x^{0'} - x_0^0$ . For  $\nu \ll 1$ , Eq. (40) can be solved for  $s$  by iteration, resulting in a power series in  $\nu$ :

$$\begin{aligned} X^{(0)} = s = \Delta x^0 + \left[ \Delta x^1 \sin\left(\frac{\Delta x^0 \omega}{c}\right) - \Delta x^2 \cos\left(\frac{\Delta x^0 \omega}{c}\right) \right] \nu - \frac{1}{2} \Delta x^0 \nu^2 \\ - \frac{1}{2a} \left[ -a + 2 \Delta x^1 \cos\left(\frac{\Delta x^0 \omega}{c}\right) + 2 \Delta x^2 \sin\left(\frac{\Delta x^0 \omega}{c}\right) \right] \left[ \Delta x^2 \cos\left(\frac{\Delta x^0 \omega}{c}\right) - \Delta x^1 \sin\left(\frac{\Delta x^0 \omega}{c}\right) \right] \nu^3 + O(\nu)^4 \end{aligned} \quad (41)$$

where I use the notation

$$\Delta x^i = x^{i'} - x_0^i \quad (42)$$

For future reference, I label the coordinates of  $P$ ,  $P'$ , and  $P_0$  in the global inertial frame by  $P' = (x^{0'}, x^{1'}, x^{2'}, x^{3'})$ ,  $P = (x^0, x^1, x^2, x^3)$ , and  $P_0 = (x_0^0, x_0^1, x_0^2, x_0^3)$ .

The contravariant spatial Fermi coordinates,  $X^{(\alpha)}$ ,  $\alpha = 1, 2, 3$ , are defined to be [12]

$$X^{(\alpha)} = \sigma \mu^i \lambda_i^{(\alpha)} = g_{ij} \sigma(s) \mu^i(s) \eta^{(\alpha\beta)} \lambda_{(\beta)}^j(s) \quad (43)$$

where I used the Minkowski definition of space-time distance,  $\sigma$ , along the space-like geodesic between  $P$  and  $P'$

$$\sigma^2 = g_{ij}(x^{i'} - x^i)(x^{j'} - x^j) \quad (44)$$

where  $g_{ij}$  is the Minkowski metric and  $\eta^{(ij)} = \eta_{(ij)} = \eta_{ij}$  is an invariant matrix, which is also equal to the Minkowski metric (given in the Appendix). The parameter  $s$  in Eq. (43) is a function of  $P'$  coordinates  $x^{i'}$ , so  $s$  must be eliminated using Eq. (41), so that  $X^{(\alpha)}$  are explicit functions of the global inertial coordinates  $x^{i'}$ . I have calculated this transformation correct to fourth order in  $\nu$ . However, since the expressions are complicated, I write them here correct only to third order in  $\nu$ :

$$\begin{aligned} X^{(1)} = \Delta x^1 - a \cos\left(\frac{\Delta x^0 \omega}{c}\right) + \frac{1}{4} \left[ \Delta x^1 - \Delta x^1 \cos\left(\frac{2\Delta x^0 \omega}{c}\right) - \Delta x^2 \sin\left(\frac{2\Delta x^0 \omega}{c}\right) \right] \nu^2 \\ + \frac{1}{2a} \Delta x^0 \left[ -\Delta x^2 + a \sin\left(\frac{\Delta x^0 \omega}{c}\right) \right] \nu^3 + O(\nu)^4 \end{aligned} \quad (45)$$

$$\begin{aligned} X^{(2)} = \Delta x^2 - a \sin\left(\frac{\Delta x^0 \omega}{c}\right) + \frac{1}{4} \left[ \Delta x^2 + \Delta x^2 \cos\left(\frac{2\Delta x^0 \omega}{c}\right) - \Delta x^1 \sin\left(\frac{2\Delta x^0 \omega}{c}\right) \right] \nu^2 \\ + \frac{1}{2a} \Delta x^0 \left[ \Delta x^1 - a \cos\left(\frac{\Delta x^0 \omega}{c}\right) \right] \nu^3 + O(\nu)^4 \end{aligned} \quad (46)$$

$$X^{(3)} = \Delta x^3 \quad (47)$$

The inverse transformation can be obtained from Eqs. (45)–(47) by iteration, resulting in

$$x^{0'} - x_0^0 = X^{(0)} + \left[ X^{(2)} \cos\left(\frac{\omega X^{(0)}}{c}\right) - X^{(1)} \sin\left(\frac{\omega X^{(0)}}{c}\right) \right] \nu + \frac{1}{2} X^{(0)} \nu^2 + O(\nu)^3 \quad (48)$$

$$x^{1'} - x_0^1 = X^{(1)} + a \cos\left(\frac{\omega X^{(0)}}{c}\right) + \frac{1}{4} \left[ X^{(1)} - X^{(1)} \cos\left(\frac{2\omega X^{(0)}}{c}\right) - X^{(2)} \sin\left(\frac{2\omega X^{(0)}}{c}\right) \right] \nu^2 + O(\nu)^3 \quad (49)$$

$$x^{2'} - x_0^2 = X^{(2)} + a \sin\left(\frac{\omega X^{(0)}}{c}\right) + \frac{1}{4} \left[ X^{(2)} + X^{(2)} \cos\left(\frac{2\omega X^{(0)}}{c}\right) - X^{(1)} \sin\left(\frac{2\omega X^{(0)}}{c}\right) \right] \nu^2 + O(\nu)^3 \quad (50)$$

$$x^{3'} - x_0^3 = X^{(3)} \quad (51)$$

## V. METRIC IN FERMI COORDINATES

The line element in the Fermi coordinate system of the observer is written as

$$ds^2 = -G_{(ij)} dX^{(i)} dX^{(j)} \quad (52)$$

where  $G_{(ij)}$  are the metric tensor components when the  $X^{(i)}$  are used as coordinates. A direct calculation of the metric tensor components from the tensor transformation rule

$$G_{(ij)} = g_{kl} \frac{\partial x^k}{\partial X^{(i)}} \frac{\partial x^l}{\partial X^{(j)}} \quad (53)$$

using the transformation Eqs. (45)–(47) and the Minkowski metric for  $g_{ij}$ , gives

$$G_{(00)} = -(1 + \zeta)^2 \quad (54)$$



and  $G_{(\alpha\beta)} = \delta_{\alpha\beta}$ , where  $\zeta$  is given by

$$\zeta = \left[ 1 - \frac{\Delta x^1}{a} \cos\left(\frac{\omega \Delta x^0}{c}\right) - \frac{\Delta x^2}{a} \sin\left(\frac{\omega \Delta x^0}{c}\right) \right] \nu^2 + O(\nu^4) \quad (55)$$

Using the transformation Eqs. (45)–(47) to express  $\zeta$  as a function of the Fermi coordinates, I obtain:

$$\zeta = -\frac{1}{a} \left[ X^{(1)} \cos\left(\frac{\omega}{c} X^{(0)}\right) + X^{(2)} \sin\left(\frac{\omega}{c} X^{(0)}\right) \right] \nu^2 + O(\nu^4) \quad (56)$$

The result in Eq. (54) agrees with Synge's general theory [12] of Fermi coordinates specialized to flat space. Synge shows that

$$\zeta = X_{(\beta)} w^{(\beta)} = \eta_{(\alpha\beta)} X^{(\alpha)} w^{(\beta)} = \eta_{(\alpha\beta)} X^{(\alpha)} w^i \lambda_i^{(\beta)} \quad (57)$$

To third order in  $\nu$ , Eq. (56) agrees with the general theory given by Synge [12]. The quantities  $X_{(\beta)} = \eta_{(\alpha\beta)} X^{(\alpha)}$  are the covariant Fermi coordinates and  $w^{(\beta)} = w^i \lambda_i^{(\beta)}$  are the components of the observer's 4-acceleration in the Fermi coordinate system. The 4-acceleration,  $w^i$ ,  $i = 0, 1, 2, 3$ , is related to the ordinary acceleration,  $a^\beta$ ,  $\beta = 1, 2, 3$ , by

$$w^i = \frac{\gamma^2}{c^2} \frac{d^2 x^i}{dt^2} = \frac{\gamma^2}{c^2} a^i \quad (58)$$

Using this relation,  $\zeta$  can be written in terms of the global inertial coordinates of  $P$  and  $P'$  as

$$\zeta = \frac{\gamma^2}{c^2} \delta_{\alpha\beta} (x^{\alpha'} - x^\alpha(s)) a^\beta \quad (59)$$

For the circular motion treated here, from Eq. (56), the three spatial components of the observer's acceleration in Fermi coordinates are given by

$$w^{(\beta)} = -\frac{\nu^2}{a} \left( \cos\left(\frac{\omega}{c} X^{(0)}\right), \sin\left(\frac{\omega}{c} X^{(0)}\right), 0 \right) \quad (60)$$

where I have dropped fourth-order terms in  $\nu$ . To third order in  $\nu$ , the observer's acceleration is just the classical centripetal acceleration for an observer moving in a circle.

The quantity  $c^2 \zeta$  is a potential (analogous to a gravitational potential) that determines the rate of proper time with respect to coordinate time in Fermi coordinates. Proper time is kept by an ideal clock. Coordinate time depends on the definition of the reference frame and coordinates used within that frame. The quantity  $\zeta$ , given in Eq. (59), depends on acceleration  $a^\beta$ , and the difference in coordinates,  $x^{\alpha'} - x^\alpha(s)$ , from the spatial origin of the Fermi coordinates. Away from the spatial origin of Fermi coordinates, the acceleration produces an effective gravitational potential field,  $c^2 \zeta$ , which leads to a change in the relation of coordinate time to proper time, through the metric components  $G_{(ij)}$ .

## VI. APPARENT BEHAVIOR OF CLOCKS IN FERMI COORDINATES

The significance of proper time is that it is the quantity that ideal clocks keep, that it may be related to time on real clocks, and that it is closely related to measurable quantities. The significance of coordinate time is that it enters into the theory. In order to analyze experiments with clocks, we must relate what is measured, i.e., proper time intervals between space-time events, to coordinate positions and times of these events.

Consider an ideal clock whose world line is given in Fermi coordinates by  $X^{(i)} = X^{(i)}(X^{(0)})$ . At coordinate times  $X_A^{(0)}$  and  $X_B^{(0)}$ , the clock is at spatial positions  $X_A^{(\alpha)}$  and  $X_B^{(\alpha)}$ , respectively. The elapsed proper time between these two events,  $\Delta\tau = \Delta s/c$ , is given by the integral along the world line of the clock

$$\Delta s = \int_A^B \left( -G_{(ij)} \frac{dX^{(i)}}{dX^{(0)}} \frac{dX^{(j)}}{dX^{(0)}} \right)^{1/2} dX^{(0)} \quad (61)$$

### A. Stationary Clock

For the case of a stationary clock, located at constant Fermi coordinate position  $X^{(\alpha)}$ , the relation between proper time  $\tau$  and coordinate time,  $X^{(0)}$ , is given by taking  $dX^{(\alpha)} = 0$  in Eq. (52). The *rate* of proper time with respect to coordinate time is then given by

$$\frac{ds}{dX^{(0)}} = 1 + \zeta(X^{(0)}, X^{(\alpha)}) \quad (62)$$

Equation (62) gives the well-known result that the rate of proper time depends on the location of the clock [42]. For the simple case of circular motion of the observer with  $\zeta$  given by Eq. (56), substitution in Eq. (62) and integration from  $X_A^{(0)}$  to  $X_B^{(0)}$  gives

$$\Delta s = X_B^{(0)} - X_A^{(0)} - \frac{a\omega}{c} \left\{ X^{(1)} \left[ \sin\left(\frac{\omega}{c} X_B^{(0)}\right) - \sin\left(\frac{\omega}{c} X_A^{(0)}\right) \right] - X^{(2)} \left[ \cos\left(\frac{\omega}{c} X_B^{(0)}\right) - \cos\left(\frac{\omega}{c} X_A^{(0)}\right) \right] \right\} \quad (63)$$

where I have dropped terms of  $O(\nu^4)$ . Equation (63) shows that there is a complicated relation between elapsed proper time on a clock located at Fermi coordinates  $X^{(1)}, X^{(2)}$ , and Fermi coordinate time  $X^{(0)}$ . For example, a clock on the  $X^{(1)}$  axis (with  $X^{(2)} = 0$ ) has a periodic time with respect to Fermi coordinate time, given by  $\Delta s = X_B^{(0)} - (a\omega/c) X^{(1)} \sin(\omega X_B^{(0)}/c)$ . The initial conditions in Eq. (63) determine complicated phase relations between proper time and coordinate time.

### B. Clock in Circular Motion

Now consider an ideal clock that is a distance  $b$  from the origin of Fermi coordinates and the clock executes circular motion in Fermi coordinates in the  $X^{(1)}\text{-}X^{(2)}$  plane. I take the world line for this motion to be given by

$$X^{(1)} = b \cos\left(\frac{\Omega}{c} X^{(0)}\right) \quad (64)$$

$$X^{(2)} = b \sin\left(\frac{\Omega}{c} X^{(0)}\right) \quad (65)$$

$$X^{(3)} = 0 \quad (66)$$

The clock moves in the  $X^{(1)}\text{-}X^{(2)}$  plane, which coincides with the plane defined by the circular motion of the origin of Fermi coordinates (see Figure 2). The situation considered here is a simplification of the problem of a GPS satellite (with a clock) orbiting the Earth, which is itself orbiting the sun. (In the case of GPS, the satellites are in orbits that are inclined at  $55^\circ$  with respect to the Earth's equator.) However, as mentioned previously, in the case of the Earth and GPS satellite, both travel on geodesics. Since we are neglecting the effects of gravity in the problem considered here, the Fermi observer and orbiting clock are both Fermi-Walker transported (e.g., with the aid of rockets).

Substitution of the satellite clock's world line into Eq. (61) leads to the following relation between elapsed proper time observed on the clock and Fermi coordinate time

$$\Delta s = \left[ 1 - \frac{1}{2} \left( \frac{b\Omega}{c} \right)^2 \right] (X_B^{(0)} - X_A^{(0)}) - \frac{ab}{c} \frac{\omega^2}{\Omega - \omega} \left[ \sin\left(\frac{\Omega - \omega}{c} X_B^{(0)}\right) - \sin\left(\frac{\Omega - \omega}{c} X_A^{(0)}\right) \right] \quad (67)$$

The *rate* of proper time with respect to coordinate time is

$$\frac{ds}{dX^{(0)}} = -\frac{1}{2} \left( \frac{b\Omega}{c} \right)^2 - \left( \frac{\omega}{c} \right)^2 ab \cos\left(\frac{\Omega - \omega}{c} X^{(0)}\right) \quad (68)$$

The proper time has a constant rate offset from coordinate time, represented by the first term in Eq. (67). In addition, the proper time has a periodic component with respect to coordinate time, given by the second term in Eq. (67).

To demonstrate the magnitude of these effects, I use parameters that correspond to Earth-Sun distance and GPS satellites. I compute the magnitudes of the following terms with the values shown in Table I:

$$\frac{1}{2} \left( \frac{b\Omega}{c} \right)^2 = 8.348 \times 10^{-11} \quad (69)$$

$$\frac{1}{2} \left( \frac{b\Omega}{c} \right)^2 \times \frac{2\pi}{\Omega} = 3.597 \times 10^{-6} \text{ sec} \quad (70)$$

Equation (69) gives the constant rate offset of the moving clock with respect to Fermi coordinate time, due to time dilation. In GPS, this term leads to a net slowing down of GPS clocks by approximately  $7 \mu\text{s}$  per day [37], which results from multiplying the value given in Eq. (70) for a single revolution by a factor of 2 (since the GPS satellites make approximately two revolutions per Earth day.)

The second term in Eq. (67) is a periodicity of the proper time with respect to Fermi coordinate time with amplitude

$$\frac{ab}{c} \frac{\omega^2}{\Omega - \omega} = 12.05 \times 10^{-9} \text{ s} \quad (71)$$

which corresponds to an amplitude in the rate

$$ab \left( \frac{\omega}{c} \right)^2 = 1.755 \times 10^{-12} \text{ s} \quad (72)$$

The proper time given by Eq. (67) is periodic with respect to Fermi coordinate time with the difference frequency,  $\Omega - \omega$ , which is the difference frequency between satellite and observer rotations. This is a kinematic effect due to the type of coordinate system used (Fermi coordinates) and the fact that the observer (Fermi coordinate frame) is moving along an arc of a circle, and therefore experiences an acceleration. Note that the periodic term vanishes when the observer angular velocity  $\omega = 0$ . If the observer were moving in a straight line and hence had zero acceleration,  $\omega^i = 0$ , the periodic effect would also be absent, as can be seen from Eq. (57).

### C. Sagnac-Like Effect for Portable Clocks

Sagnac [38–40] demonstrated that there exists a phase shift between two counter propagating light beams on a rotating platform and that this phase shift depends on the angular frequency of rotation of the platform. In the Fermi coordinates of the observer moving in a circle, there exists a Sagnac-like effect for two portable clocks that are orbiting in opposite directions.

Consider a clock that moves eastward along a circle of radius  $b$  in the equatorial plane of the Fermi coordinates with world line given by Eqs. (64)–(66) (see Figure 2). The speed of the clock in Fermi coordinates is  $b\Omega/c$ . Assuming that the clock begins its journey at point  $A$  at time  $X_A^{(0)} = 0$  and ends it at  $X_B^{(0)} = 2\pi c/\Omega$ , which corresponds to one revolution, the proper time that elapses on the clock is given by Eq. (67) as

$$\Delta s_+ = \left[ 1 - \frac{1}{2} \left( \frac{b\Omega}{c} \right)^2 \right] \frac{2\pi}{\Omega} c + \frac{ab}{c} \frac{\omega^2}{\Omega - \omega} \sin \left( 2\pi \frac{\omega}{\Omega} \right) \quad (73)$$

The first term in Eq. (73) is the time dilation due to the speed  $b\Omega/c$  of the clock. The second term depends on the path traversed.

Next, consider an identical clock that starts at point  $A$  at time  $X_A^{(0)} = 0$  but moves westward. The world line of this clock is then given by Eq. (64)–(66) but with  $\Omega = -|\Omega|$ . The proper time on this clock is given by

$$\Delta s_- = \left[ 1 - \frac{1}{2} \left( \frac{b\Omega}{c} \right)^2 \right] \frac{2\pi}{|\Omega|} c - \frac{ab}{c} \frac{\omega^2}{|\Omega| + \omega} \sin \left( 2\pi \frac{\omega}{|\Omega|} \right) \quad (74)$$

When the east-moving and west-moving clocks have each made one revolution and they have returned to their starting positions at point  $A$ , the difference in their times is given by  $\Delta\tau_+ - \Delta\tau_- = (\Delta s_+ - \Delta s_-)/c$ . For the case applicable to GPS,  $b \ll a$  and  $\omega \ll |\Omega|$ , the difference of proper time on the two clocks is given by

$$\tau_+ - \tau_- = 2 \frac{b}{a} \left( \frac{a\omega}{c} \right)^2 \frac{|\Omega|}{\Omega^2 - \omega^2} \sin \left( 2\pi \frac{\omega}{|\Omega|} \right) \quad (75)$$

$$= 4\pi \frac{b}{a} \left( \frac{a\omega}{c} \right)^2 \frac{\omega}{\Omega^2} + O\left(\frac{\omega^2}{\Omega^2}\right) \quad (76)$$

where in the last line I have dropped terms of  $O(\omega^2/\Omega^2)$ . This Sagnac-type effect is due to the non-inertial nature of the Fermi coordinate system. The origin of this effect can be understood from the point of view of the global inertial

coordinate system. If the Fermi coordinate origin moved along a straight line in Minkowski space (rather than a circle), then there would be no acceleration and  $\zeta = 0$ . In this case, the east-moving and west-moving clocks would have zero time difference after one revolution. However, since the origin of the Fermi coordinates moves along a circle, the acceleration picks out a direction in the Fermi coordinate space-time, and the symmetry between the two clock world lines is broken. Consequently, the elapsed proper time is different on the two clocks when they are compared after one revolution. Note that the time difference on the two clocks increases with increasing angular velocity of rotation  $\omega$ , analogous to the optical Sagnac effect [38–40]. This increase is due to the fact that the breaking in symmetry is due to the magnitude of the acceleration,  $w^{(\beta)}$ , see Eq. (57). Using the values of the parameters appropriate to GPS satellites, given in Table 1, I find the magnitude of the time difference on the two clocks after one revolution to be

$$\tau_+ - \tau_- = 2.066 \times 10^{-10} \text{ s} \quad (77)$$

This is a small effect that vanishes when the origin of Fermi coordinates moves along a straight line or when  $\omega = 0$ .

## VII. COORDINATE SPEED OF LIGHT

The three-dimensional space of Minkowski space-time is homogeneous and isotropic, and consequently, the speed of light is a constant. However, for the case of Fermi coordinates of an observer moving in a circle, the symmetry of the 3-d space is broken by an acceleration. Consequently, the space is not homogeneous and isotropic, and the coordinate speed of light depends on position in the 3-d space.

For a general space-time metric  $g_{ij}$ , the coordinate speed of light  $v_c$  in the direction of a unit vector  $e^\alpha$  is given by [41]

$$v_c = \frac{\sqrt{-g_{00}}}{1 + \gamma_\alpha e^\alpha} c \quad (78)$$

where

$$\gamma_\alpha = \frac{g_{0\alpha}}{\sqrt{-g_{00}}} \quad (79)$$

and the unit vector satisfies the relation  $\gamma_{\alpha\beta} e^\alpha e^\beta = 1$ , where the 3-d spatial metric is given by [42]

$$\gamma_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha} g_{0\beta}}{g_{00}} \quad (80)$$

Equation (78) shows that for a general space-time metric the speed of light depends on direction  $e^\alpha$  and on position and time, through the metric components. A non-zero off-diagonal metric component  $g_{0\alpha}$  leads to an anisotropy (directional dependence) of the coordinate speed of light. The metric for Fermi coordinates given in Eq. (54) has vanishing off-diagonal terms,  $\gamma_\alpha = 0$ , and consequently the coordinate speed of light is isotropic. However, the metric component  $G_{(00)}$  has a non-trivial position and time dependence, and Eq. (78) gives the coordinate speed of light:

$$v_c = c(1 + \zeta) \quad (81)$$

where  $\zeta$  is a function of Fermi coordinate position and time, given by Eq. (56). Therefore, I can write the coordinate dependence of the speed of light as

$$\frac{\Delta c}{c} = -\frac{\nu^2}{a} \left[ X^{(1)} \cos\left(\frac{\omega}{c} X^{(0)}\right) + X^{(2)} \sin\left(\frac{\omega}{c} X^{(0)}\right) \right] + O(\nu^4) \quad (82)$$

This variable speed of light is a kinematic effect due to the non-inertial nature of the Fermi coordinate system. Note that at any given position  $\mathbf{X} = \{X^{(1)}, X^{(2)}, X^{(3)}\}$  the speed of light depends on time in a periodic way, with the orbital period of the reference frame,  $\omega$ . This expression can be rewritten as

$$\frac{\Delta c}{c} = a \left(\frac{\omega}{c}\right)^2 \mathbf{X} \cdot \hat{\mathbf{n}} + O(\nu^4) \quad (83)$$

where the unit vector  $\hat{\mathbf{n}} = (-\cos(\frac{\omega}{c} X^{(0)}), -\sin(\frac{\omega}{c} X^{(0)}), 0)$  is in the observer's orbital plane and points in the direction of acceleration, toward the center of the circle (see Figure 3). As the Fermi reference frame moves around

the circle, the vector  $\hat{n}$  always points toward the circle's center in Fermi coordinates. At any given time  $X^{(0)}$ , Eq. (83) shows that to second order in  $\nu$ , in the plane given by

$$\mathbf{X} \cdot \hat{n} = 0 \quad (84)$$

the speed of light does not differ from its vacuum inertial frame value  $c$ . In the 3-d half-space containing positions  $\mathbf{X}$  closer to the center of the circle, the speed of light is increased. For the 3-d half-space containing positions  $\mathbf{X}$  that are further from the center of the circle, the speed of light is decreased. As time  $X^{(0)}$  increases, the dividing plane, given by Eq. (84), rotates at angular frequency  $\omega$  around the circle, with the origin of Fermi coordinates remaining the point at which the plane is tangent to the circle.

The magnitude of the inhomogeneous variation in the speed of light in Eq. (82) is small. For the kinematics of GPS, when I use the numbers given in Table 1, with the coordinates  $X^{(1)}, X^{(2)} \sim b$ , so the effect has a magnitude

$$\frac{\nu^2}{a} b = 1.755 \times 10^{-12} \quad (85)$$

and has a complicated position and time dependence, given by Eq. (82). Recently, GPS data have been used to obtain an upper bound on the *anisotropy* of the speed of light:  $\Delta c/c < 10^{-9}$  (see Wolf and Petit [43] and references contained therein). Given the present accuracy of GPS, the small inhomogeneity of the speed of light given by Eq. (82), is probably not measurable.

### VIII. SUMMARY

I have considered the problem of an observer moving in a circle around a central point in flat space (no gravitational fields). The observer carries with him three gyroscopes and uses them to define a set of non-rotating axes. He keeps his axes at a constant orientation with respect to the gyroscopes' spin axes. The observer's axes are Fermi transported along his world line and Fermi coordinates are defined. The Fermi time coordinate is just the observer's proper time, according to the construction shown in Figure 1. Fermi coordinates play the role of ECI frame coordinates in GPS. I investigate the kinematic effects seen on ideal clocks from the point of view of these Fermi coordinates. Specifically, I consider two cases: stationary clocks (with respect to Fermi coordinates) and satellite clocks in circular orbit with respect to Fermi coordinates. The proper time interval between two events on the stationary clock, given by Eq. (63), is a complicated periodic function of the clock's Fermi coordinate time and coordinate position in the plane perpendicular to the rotation axis. I considered next a clock that orbits the Fermi coordinate origin (similar to a satellite) in the equatorial plane of rotation and computed the relation of proper time to coordinate time. I find that the proper time on the clock has time dilation (constant rate offset) plus a periodic dependence on Fermi coordinate time at the difference frequency,  $\Omega - \omega$ , between satellite clock orbital frequency,  $\Omega$ , and Fermi coordinate frame orbital frequency,  $\omega$ .

Next, I looked at the time difference of two counter-orbiting clocks in the equatorial plane of rotation, one west-moving and the other east-moving. After one revolution, the difference in proper time on the clocks shows a small Sagnac-like effect, increasing with the angular frequency of revolution of the Fermi frame,  $\omega$ , given by Eq. (76).

Finally, I computed the coordinate speed of light in the Fermi coordinates. The speed of light is isotropic (the same in all directions), but is not homogeneous. Specifically, the speed of light depends on Fermi coordinate position and time, but not on direction. Equation (83) gives the change in the speed of light from  $c$ , its vacuum value in an inertial frame.

I have computed the magnitudes of these effects for numbers relevant to GPS satellite orbits and orbital frequency. The effects are all small for GPS, except for the 12 ns periodicity for an orbiting clock, given in Eq. (67), that occurs at the difference frequency,  $\Omega - \omega$ , between satellite orbital frequency  $\Omega$  and Earth orbital frequency  $\omega$ . This effect is reminiscent of the periodic effect in GPS described in the introduction. However, the effects here are all kinematic, since gravitational fields are neglected. The inclusion of gravitational fields is expected to significantly modify the results. In particular, from the work of Ashby and Bertotti, this 12 ns periodic effect is expected to cancel when gravitational fields are included [2,28]. Work is in progress to extend these results to include the effects of gravitational fields.

While the effects discussed in this work are small for the present GPS, these effects are quite significant for future navigation and time transfer systems that are expected to have sub-nanosecond time transfer accuracy [6–8].

### ACKNOWLEDGMENTS

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## APPENDIX A: CONVENTIONS AND NOTATION

I use the notation and conventions of Synge [12], except that my space-time indices on coordinates,  $x^i$ , take values  $i = 0, 1, 2, 3$ . Specifically, I use the convention that Roman indices take the values  $i = 0, 1, 2, 3$ , and Greek indices  $\alpha = 1, 2, 3$ . I use the summation convention for repeated indices: when the same index appears in a lower and upper position, summation is implied over the range of the index.

If  $x^i$  and  $x^i + dx^i$  are two events along the world line of an ideal clock, then the square of the proper time interval between these events is  $d\tau = ds/c$ , where  $ds$  is given in terms of the space-time metric as

$$ds^2 = -g_{ij} dx^i dx^j \quad (A1)$$

and I choose  $g_{ij}$  to have the signature +2. When  $g_{ij}$  is diagonalized at any given space-time point, the elements can take the form of the flat space Minkowski metric given by  $g_{00} = -1$ ,  $g_{\alpha\beta} = \delta_{\alpha\beta}$ , and  $g_{0\alpha} = 0$ .

In this work, I take the global inertial frame coordinates to have the Minkowski metric. The observer's frame is described by Fermi coordinates,  $X^{(a)}$ ,  $a = 0, 1, 2, 3$ , with line element

$$ds^2 = -G_{(ij)} dX^{(i)} dX^{(j)} \quad (A2)$$

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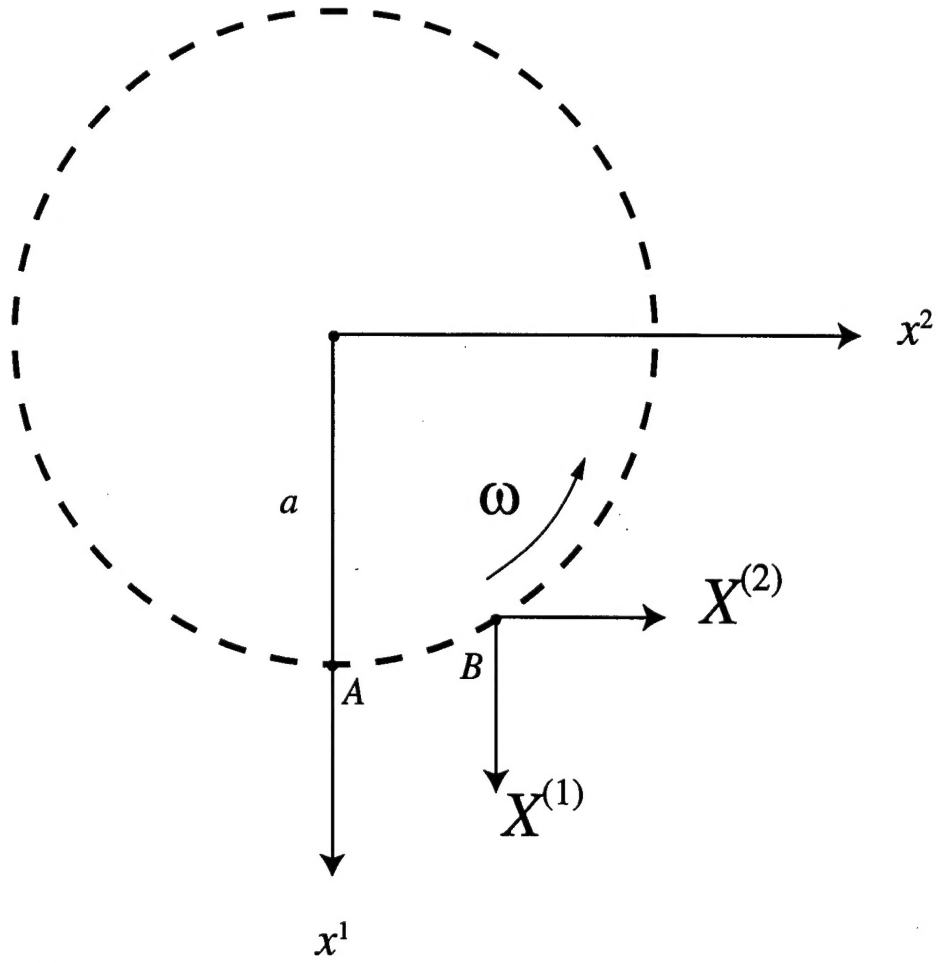


FIG. 1. The observer's circular 3-d path is shown against the background of the global inertial Minkowski coordinates  $x^i$ . The position of the observer (origin of Fermi coordinates) and the orientation of the Fermi coordinate axes  $X^{(a)}$  are shown schematically at a given time.



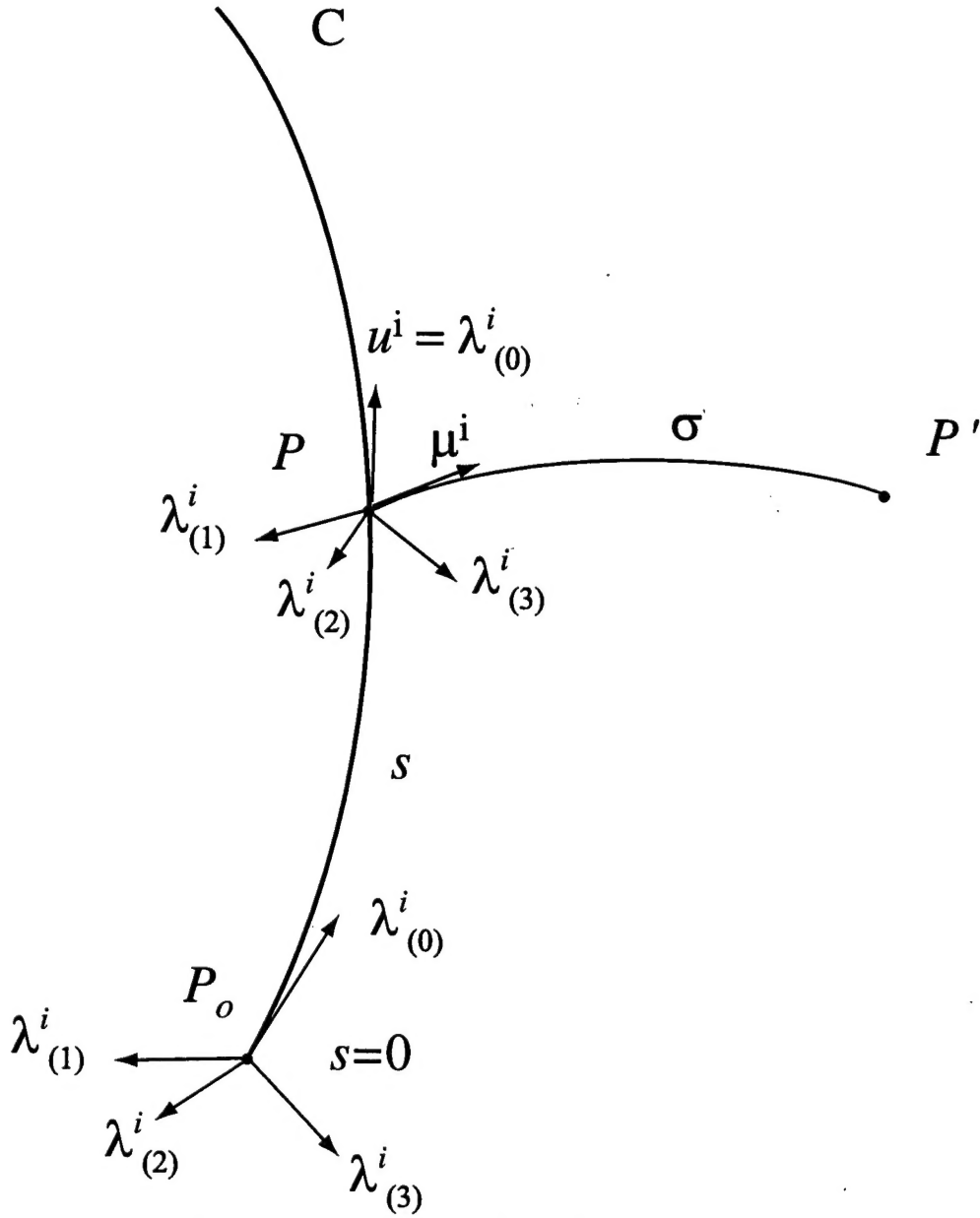


FIG. 2. The observer's world line  $C$  is shown with the initial tetrad basis vectors  $\lambda^i_{\alpha}$  at  $s = 0$  at point  $P_0$ . The Fermi transported tetrad basis vectors at finite proper time  $s$  are shown at point  $P$ .

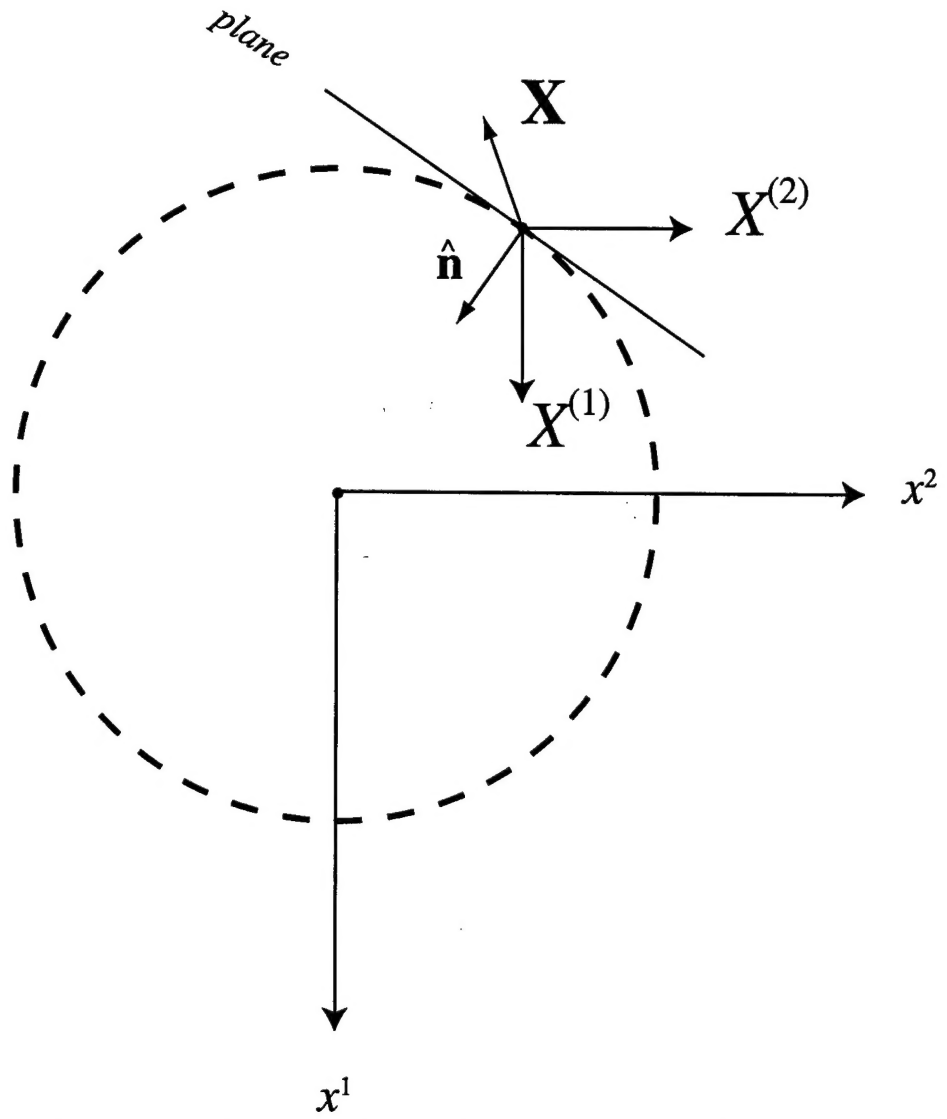


FIG. 3. The global inertial coordinate axes  $x^i$  and Fermi coordinates  $X^{(a)}$  are shown in relation to the coordinate position  $\mathbf{X}$ , unit vector  $\hat{\mathbf{n}}$ , and the plane  $\mathbf{X} \cdot \hat{\mathbf{n}} = 0$ , at a given coordinate time  $X^{(0)}$ .

TABLE I. Numerical values of constants.

constant	definition	value
$a$	Earth orbital semi-major axis	$1.496 \times 10^{11} \text{ m}$
$\omega$	Earth orbital angular velocity	$1.99238 \times 10^{-7} \text{ sec}^{-1}$
$b$	GPS satellite semi-major axis	$2.656177 \times 10^7 \text{ m}$
$\Omega$	GPS satellite angular velocity	$1.45842 \times 10^{-4} \text{ sec}^{-1}$
$c$	vacuum speed of light	$2.997924 \times 10^8 \text{ m/s}$